Covariance Evaluation by Means of Uncertainty Assessment

Christophe Dubois, Laurent Leblond, Jean-Michel Pou, and Alessandro Ferrero

One of the key points in modern measurement science is the evaluation of measurement uncertainty, according to the definition given by the International Vocabulary of Metrology (VIM) [1] and the procedure recommended by the *Guide to the Expression of Uncertainty in Measurement* (GUM) and its supplements [2]–[4]. According to these documents, when several sources of uncertainty are present, the related standard uncertainties shall be quadratically combined, and the covariances must be considered, if present [2].

Several examples can be found in the literature, where covariances play a significant role in the evaluation of the combined standard uncertainty, including, but not limited to, interlaboratory comparisons [5], system identification [6], sensor data fusion [7], and vision-based measurements [8]. These same examples show that covariances are not always easy to detect and evaluate, and strict mathematical methods to evaluate them, although available [9], [10], are often complex to apply and time consuming. For this reason, covariances are often neglected or very roughly estimated especially in industrial measurement processes. Thus, ineffective and unreliable uncertainty estimation may be obtained.

This article is aimed at showing that a careful analysis of the uncertainty budget of a measurement process allows us to obtain a reasonable estimate of – at least – the order of magnitude of the covariances, even when their experimental or strict theoretical evaluation is impractical. In this respect, it proposes a type B method for covariance estimation that extends the method presented by [11] and considers an error decomposition model based on the error variation rate in the measurement process.

Background

It is well known that a measurement process cannot return the true value of the measurand that is the value of the measured quantity, due to the presence of measurement errors that are generally unknown and unknowable, and are caused by a number of **influence quantities**.

According to the recent development of measurement science, a measurement result cannot be expressed by a single value but by "*a set of quantity values being attributed to a measurand together with any other available relevant information*" [1], where this information "may be expressed in the form of a probability density function" [1]. It can be then stated that a measurement result, hence the knowledge we have about the measurand, can be represented by a real random variable X [12]. (From a strict mathematical point of view, this random variable must be also quadratically integrable on a probability space.) Every realization of this variable can be considered as a value returned by the measurement process.

According to the GUM, X is characterized by a mean value $\mu_X = \mathbb{E}(X)$ and a standard deviation (called in the measurement field *standard uncertainty* [1], [2]) $\sigma_X = \sqrt{\mathbb{V}(X)}$ [2]. Operator $\mathbb{E}$ is the mathematical *expectation*, and operator $\mathbb{V}$ is the *variance*.

An interesting problem arises when a generic quantity Y is generally dependent on k other quantities $\underline{X} = (X_1,\ldots,X_k)$, that can be measured by the same or different measurement processes and are affected by a number of influence quantities. Let us suppose that Y can be evaluated through a known **measurement model** f with a possible model error ϵ with $\epsilon(\mu_\epsilon) = 0$, independent of any X_i, $\forall i$, $1 \le i \le k$. It is:

$$Y = f(\underline{X}) + \epsilon \tag{1}$$

The correct expression of the measurement result of Y requires (1) to provide both μ_Y– considered as the maximum likelihood value of the measurand – and the standard uncertainty σ_Y. Since Y is obtained through the measurement of X, μ_Y and σ_Y should be obtained from μ_{X_i} and σ_{X_i}. The well-known solution, proposed by the GUM [2], is based on the first-order expansion of f about $\mu_{\underline{X}} = (\mu_{X_1},\ldots,\mu_{X_k})$:

$$f(\underline{X}) \approx f(\mu_{\underline{X}}) + \frac{\partial f}{\partial X_1}(\mu_{\underline{X}})(X - \mu_{X_1}) + \ldots \frac{\partial f}{\partial X_k}(\mu_{\underline{X}})(X - \mu_{X_k}) \tag{2}$$

According to this approximation, expectation and variance can be computed as:

1094-6969/16/$25.00©2016IEEE

$$\mu_Y \approx f(\mu_{\underline{X}}) \tag{3}$$

$$\sigma_Y^2 \approx \sum_{i=1}^{k}\left(\frac{\partial f}{\partial X_i}(\mu_{\underline{X}})\sigma_{X_i}\right)^2 + \sum_{\substack{i,j=1\\ i\neq j}}^{k}\frac{\partial f}{\partial X_i}(\mu_{\underline{X}})\cdot\frac{\partial f}{\partial X_j}(\mu_{\underline{X}})\cdot\sigma_{X_i,X_j} + \sigma_\epsilon^2 \tag{4}$$

To evaluate (4), and, hence, the combined standard uncertainty, the covariance $\sigma_{X_i,X_j} = \mathbb{C}ov(X_i,X_j)$ must be evaluated. A covariance expresses the simultaneous (correlated) variation of the measurement errors related to two quantities. It is positive when the variation of one variable implies a variation in the same direction of the other variable. It is negative when the variation of the other variable is in the opposite direction.

Covariance models mathematically the effects that are always unknown on measurement uncertainty of the influence quantities that are in common, either totally or partially, with X_i and X_j. As stated in the introduction, its evaluation is not easy, especially when the considered effects are not totally correlated.

A possible, simpler way to estimate covariances is based on the analysis of the **physical properties** of the influence quantities and the way they affect measurement uncertainty. Noise, for instance, can be often considered as a common source of uncertainty. However, its variation rate is so high during a measurement process that its effects on X_i and X_j may be considered totally independent, so that in this case, $\sigma_{X_i,X_j} = 0$. From this perspective, it is possible to state that noise has a *High Opportunity* (HO) to express its variations during the measurement process.

On the other hand, temperature is a common source of uncertainty that does not generally change during the measurement process. It may remain constant, or change in such a way that it affects X_i and X_j in the same way, as in the first example in the next section, or it may affect X_i and X_j in a slightly different way, as in the second example of the next section.

In all such cases, different from the case involving noise, it is possible to state that this common influence quantity has a *Low Opportunity* (LO) to express its variations during the measurement process. Therefore, covariance is expected to take non-zero values only due to the effects of LO influence quantities.

It can be proved (Appendix A) that covariances can be estimated by the following formula:

$$\sigma_{X_i,X_j} = \sigma_{X_i}\ \sigma_{X_j}\sum_{l-1}^{m} \pm_l^{i,j}\gamma_l\sqrt{\omega_i,l\ \omega_j,l} \tag{5}$$

where each influence quantity l, l = 1…m is considered, $\pm_l^{i,j} = 1$ if the influence quantity affects both X_i and X_j in the same direction, and $\pm_l^{i,j} = -1$ in the opposite case. Moreover, γ_l is the fraction of LO variance of l on X_i and X_j (Appendix A), and $w_{i,l}$ and $w_{i,j}$ are the relative weights with which the influence quantity l affects the total uncertainty on X_i and X_j, respectively.

Two Application Examples

To better understand the proposed simplified method, two simple examples are provided. The first one is related to thickness measurement of a mechanical part, and the second one to the measurement of an electric current.

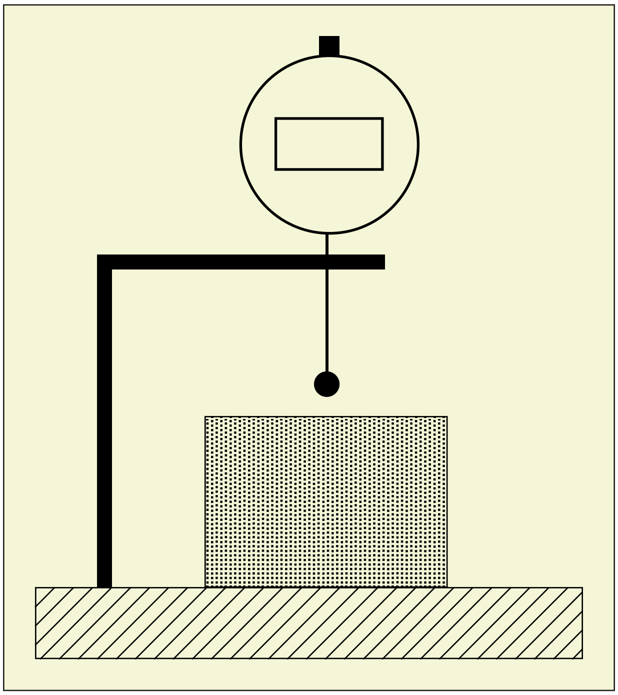

Fig. 1. Thickness measurement by means of a comparator.

Averaging Two Measurements of Mechanical Parts' Thicknesses

In this example, the measurement of the thickness of a mechanical part is considered. The thickness is measured at two different points, and the two obtained values are averaged to obtain the final measured value that, in the considered example, is supposed to be μ_X = 10 mm. Fig. 1 shows the measurement set-up and is simply composed by a marble reference plane and a comparator. A gauge block with L = 10 mm is used as the comparator reference.

Assuming that the measured thickness values at the measurement points a and b are, respectively, X_a and X_b, the quantity of interest is hence:

$$\overline{X} = \frac{X_a + X_b}{2} \tag{6}$$

The following sources of uncertainty affect the thickness measurement and are supposed to be independent:

- *Repeatability*: repeatability of the measurement process experimentally evaluated
- *Resolution*: comparator resolution of 0.001 mm, already included in the repeatability
- *Reference*: calibration uncertainty of the gauge block used as reference
- *Operator*: uncertainty contribution given by operator's lack of skill in performing the measurement, experimentally evaluated
- *Part:* the part under measurement is not perfectly flat, and a flatness estimation can be given with a tolerance of 1 µm

Table 1 – Uncertainty budget							
Classical uncertainty budget						**Additional terms**	
N°	*Source* (l)	*Type*	*Half interval* (μm)	*K*	u_l (μm)	*Weight* (w_l)	*LO Part* (γ_l)
1	Repeatability	A	-	-	0.70	24%	0%
2	Reference	B	$0.14 + 2 \times 10^{-6}$ L	2	0.08	0%	Negligible
3	Operator	A	-	-	1.10	59%	100%
4	Part	B	1	$\sqrt{3}$	0.58	17%	0%

Starting from the above assumption, the data in Table 1 can be obtained. The left side shows a classical uncertainty budget obtained following the classical GUM approach [2], and the right side shows some additional information, as discussed below. The conventional notation is used, and symbol u_l is used for the standard uncertainty associated with each influence quantity l, instead of symbol σ_l, generally used for standard deviations. The method (type A or type B) followed to evaluate the considered uncertainty contributions is also reported. When a type B method is followed, the estimated half interval for the expanded uncertainty is reported, as well as the assumed coverage factor K [2], to obtain the standard uncertainty value from the half interval.

The combined standard uncertainty on a single thickness measurement result X can be readily obtained by quadratically adding the u_l values in Table 1:

$$u_X = \sqrt{\sum_l u_l^2} \simeq 1.43\,\mu\text{m} \tag{7}$$

The standard uncertainty of the average of two measurement results can be strictly evaluated (Appendix A) taking into account covariance σ_{X_a, X_b} among X_a and X_b, since these two variables cannot be considered as independent, due to the fact that the same operator performs both measurements. Under this assumption, it is:

$$u_{\bar{X}} = \sqrt{\frac{u_{X_a}^2 + u_{X_b}^2 + 2\sigma_{X_a X_b}}{4}} \tag{8}$$

Since the strict evaluation of σ_{X_a, X_b}, and, consequently, of (8) can be quite difficult, correlation is not generally considered, and the standard uncertainty is evaluated as if the two measurement results were independent, thus obtaining:

$$u_{\bar{X}}^{\text{without covariance}} = \frac{u_X}{\sqrt{2}} \simeq 1.01\,\mu\text{m} \tag{9}$$

The proposed simplified method, on the other hand, allows one to estimate σ_{X_a, X_b} by completing the classical uncertainty budget. This can be done by exploiting the information already present in the classical budget to identify the LO contributions to be used in (5), as Appendix A shows:

- the first column in the right part of Table 1 shows the weight $w_l = \frac{u_l^2}{\sum_{k=1}^{m} u_k^2}$ with which every factor l contributes to the final uncertainty and
- the second column includes the LO part γ_l for each influence quantity l.

The determination of the LO part is based on the available information and *a priori* experience on the measurement process. In the considered example, the uncertainty contributions related to repeatability and the measured part are independent among the two measurements. Therefore, the LO part of both contributions is zero by definition. On the other hand, since the same operator is executing both measurements, its contribution to measurement uncertainty is the same on both results, and consequently, the appertaining LO part is $\gamma_{Op} = 1$ (100%). At last, since the weight of the reference contribution on the final uncertainty has been considered negligible, the related LO part is also not considered, thus simplifying the computation.

Since, by assumption, the operator – which represents the only LO factor – affects both measurements in the same way, it is:

$$\pm_{\text{Op}}^{a,b} = 1,$$

and hence, (5) results in:

$$\sigma_{X_a, X_b} = u_{X_a} u_{X_b}\, \gamma_{Op.} \sqrt{w_{Op.} w_{Op.}} = 1.21\,\mu\text{m}^2 \tag{10}$$

Since the measurement process is the same for the two measurements, $u_{X_a} = u_{X_b} = u_X$ is assumed. Therefore, the values provided by (7) and (10), used in (8), give the standard uncertainty on the average value of the two measurements, as:

$$u_{\bar{X}} = \frac{1}{\sqrt{2}} \cdot \sqrt{u_X^2 + \sigma_{X_a, X_b}} = \frac{u_X}{\sqrt{2}} \cdot \sqrt{1 + w_{Op.}} \simeq 1.28\,\mu\text{m} \tag{11}$$

This result shows that uncertainty underestimation can be significant if a covariance is ignored. If $E_{a,op}$ and $E_{b,op}$ are the errors made by the operator on points a and b respectively, the correlation coefficient between these two errors is 1 by assumption:

$$r_{E_{a,Op.}, E_{b,Op.}} = \frac{\sigma_{E_{a,Op.}, E_{b,Op.}}}{u_{E_{a,Op.}} u_{E_{b,Op.}}} = 1 \tag{12}$$

Since,

$$u_{E_{a,Op}} = \sqrt{w_{op}} \,.\, u_{X_a}, \qquad u_{E_{b,Op}} = \sqrt{w_{op}} \,.\, u_{X_b}$$

and

$$\sigma_{X_a, X_b} = \sigma_{E_{a,Op.}, E_{b,Op.}}$$

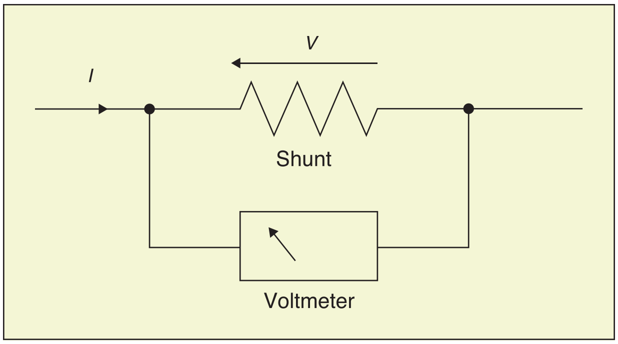

Fig. 2. Electric current measurement by means of a shunt and a digital voltmeter.

it is possible to find the same result as the one in (11) by considering total correlation. This is the procedure followed by the GUM [2] for this particular case.

Electric Current Measurement

The previous example presented a case of total correlation, that can be easily handled following the GUM [2] recommendations. However, several cases exist, such as the one considered in this example, where correlation is not total. In such cases, the GUM recommends an experimental evaluation of covariances, which is not always possible. This example shows that the proposed method can be an effective alternative to the experimental evaluation.

The quantity to be measured is an electrical current, in this example. The measurement procedure considers a shunt and a digital voltmeter, as Fig. 2 shows. The current's expected value is $\mu_I = 10$ A, the shunt has a rated value $\mu_R = 0.01\ \Omega$, and the measurement procedure is performed at room temperature $\mu_T = 20$ °C (± 3 °C).

If *V* is the measured voltage and *R* the shunt resistance, the current value *I* is obtained as:

$$I = \frac{V}{R} \tag{13}$$

The uncertainty propagation law in the GUM [2] provides the combined standard uncertainty on the measured current, having assumed $\mu_V = \mu_R\mu_I = 0.1$ V and $\underline{\mu} = (\mu_V, \mu_R)$, as:

$$u_I^2 \approx \left(\frac{\partial I}{\partial V}(\underline{\mu}).u_V\right)^2 + \left(\frac{\partial I}{\partial R}(\underline{\mu}).u_R\right)^2 + 2.\frac{\partial I}{\partial V}(\underline{\mu}).\frac{\partial I}{\partial R}(\underline{\mu}).\sigma_{V,R} \tag{14}$$

where

$$\frac{\partial \mathrm{I}}{\partial \mathrm{V}}(\underline{\mu}) = \frac{1}{\mu_\mathrm{R}} = 100\ \Omega^{-1}$$

and

$$\frac{\partial \mathrm{I}}{\partial \mathrm{R}}(\underline{\mu}) = -\frac{\mu_\mathrm{V}}{\mu_\mathrm{R}^2} = -1000\ \mathrm{A}\cdot\Omega^{-1}$$

In this example, voltage and resistance measurements can be assumed to depend on the same influence quantity: temperature. The independent uncertainty contributions affecting voltage measurement are:

- *Repeatability*: repeatability of the measurement procedure, experimentally obtained
- *Resolution*: voltmeter resolution, assumed to be 0.01 mV, already considered in the repeatability
- *Manufacturer V*: technical specifications provided by the voltmeter manufacturer (0.1 mV)
- *Temperature V*: sensitivity to temperature variations (4×10^{-5} V/°C)

The independent uncertainty contributions affecting the shunt resistance value are:

- *Manufacturer R:* technical specifications provided by the shunt manufacturer (6μΩ)
- *Temperature R:* sensitivity to temperature variations ($8 \times 10^{-7}\ \Omega$/°C)

Similar to the previous example starting from these data, it is possible to define Table 2 with the additional information that is discussed below.

From Table 2, it is possible to evaluate the combined standard uncertainty for voltage and resistance as:

$$u_V \simeq 133.5\times10^{-6}\,\mathrm{V} \quad \text{and} \quad u_\mathrm{R} \simeq 3.45\times10^{-6}\Omega \tag{15}$$

If covariance is neglected in (14), regardless of the fact that room temperature affects both the measured value of *V* and the resistance of the shunt *R*, the following result would have been obtained:

$$u_I^{without\ covariance} \approx \sqrt{\left(\frac{\partial I}{\partial V}(\underline{\mu})\cdot u_V\right)^2 + \left(\frac{\partial I}{\partial R}(\underline{\mu})\cdot u_R\right)^2} \simeq 13.8\ \mathrm{mA} \tag{16}$$

Similar to the example shown in the previous section, the additional terms considered in the uncertainty budget (Table 2) are useful to estimate the existing covariance. At measurement time, the shunt and the voltmeter are operating more or less at the same room temperature. This *more or less* implication can be quantified by means of the following practical considerations: the possible deviation between the voltmeter and shunt temperature can be assumed to be 1 °C on a possible variation range of 6 °C. Consequently, the LO part can be estimated to be: $\gamma_T \simeq \frac{5}{6} \simeq 85\%$.

Sign $\pm_T^{V,R}$ can be derived from the trend of variation of the uncertainty contributions as temperature varies. In this example, it is reasonable to consider that the errors on the voltmeter and shunt resistance increase with temperature, and consequently, $\pm_T^{V,R} = 1$ is considered. Of course, nothing changes from the theoretical point of view, if opposite variations were assumed, so that $\pm_T^{V,R} = -1$ would have been chosen. Moreover, according to Table 2, the weights $w_{V,T}$ and $w_{R,T}$ are evaluated to be $w_{V,T} = 41\%$ and $w_{R,T} = 24\%$, respectively. Therefore, (5) yields:

$$\sigma_{V,R} = u_V u_R \gamma_T \sqrt{w_{V,T} w_{R,T}} \simeq 1.23 \times 10^{-10}\ \mathrm{V}^2\Omega^2 \tag{17}$$

Table 2 – Uncertainty budget							
Classical uncertainty budget *Voltmeter*						**Additional terms**	
N°	*Source (l)*	*Type*	*Half interval*	*K*	$u_{V,l}$ (μV)	*Weight* ($w_{V,l}$)	*LO part* (γ_l)
V1	Repeatability	A	-	-	90	45%	0%
V3	Manufacturer	B	0.1 mV	2	50	14%	0%
V4	Temperature	B	0.04 mV / °C (± 3°C)	$\sqrt{2}$	85	41%	85%
Classical uncertainty budget *Resistance*						**Additional terms**	
N°	*Source (l)*	*Type*	*Half interval*	*K*	$u_{R,l}$ ($\mu\Omega$)	*Weight* ($w_{R,l}$)	*LO part* (γ_l)
R1	Manufacturer	B	6 μΩ	2	3	76%	0%
R3	Temperature	B	0.8 μΩ / °C (± 3°C)	$\sqrt{2}$	1.7	24%	85%

Considering this new value, (14) provides the combined standard uncertainty on the measured current as:

$$u_I \simeq 14.7 \text{ mA} \tag{18}$$

The comparison between (18) and (16) shows that the impact of covariance on the evaluation of the standard uncertainty on the measured value of the current is 6.5% when it is caused by a temperature influence of 40% on voltage uncertainty and 24% on resistance uncertainty. This result, that is not immediate nor intuitive, has been obtained by means of a simple analysis of the available information and simple computations.

Conclusion

The application of the uncertainty propagation law to the evaluation of measurement uncertainty, as suggested by the GUM [2], requires one to consider the covariances between the measurement errors on the input quantities. The type A evaluation of the different covariances might become quite difficult, considering the number of measurements of a perfectly stable measurand that are needed. Also, the estimation of a correlation coefficient, as proposed by the GUM, suffers from the same problems, being directly related to the covariance.

The method proposed in this article allows one to estimate the covariances starting from a physical analysis of the measurement conditions and the available relevant information. It was proved that taking into account the HO or LO characteristics of the different uncertainty contributions and the common sources of uncertainty is sufficient to estimate the covariances, under the given assumptions. This confirms that the proposed method for the evaluation of covariances can be considered as a type B method.

The two proposed examples have shown that, since this analysis (HO, LO and common sources) is easily accessible, the proposed solution offers practitioners a relatively simple method to consider all terms involved in the propagation of uncertainty and, therefore, improves the knowledge of the uncertainties involved in the measurement process. The proposed method looks interesting in industrial metrology, where the experimental evaluation of covariances, if possible, is generally time consuming and, therefore, too expensive. This method provides a time and cost effective way to estimate covariances, thus allowing a more accurate evaluation of uncertainty that is definitely more accurate than that obtained by neglecting correlations, or conversely, considering perfectly correlated measurement results.

References

[1] JCGM 200:2012, *International Vocabulary of Metrology- Basic and General Concepts and Associated Terms* (VIM 2008 with minor corrections), Joint Committee for Guides in Metrology, 2012. [Online]. Available: http: //www.bipm.org/en/publications/guides/vim.html.

[2] JCGM 100:2008, *Evaluation of Measurement Data- Guide to the Expression of Uncertainty in Measurement* (GUM 1995 with minor corrections), Joint Committee for Guides in Metrology, 2008. [Online]. Available: http: //www.bipm.org/en/publications/guides/gum.html.

[3] JCGM 101:2008, *Evaluation of Measurement Data- Supplement 1 to the Guide to the Expression of Uncertainty in Measurement- Propagation of distributions using a Monte Carlo method*, Joint Committee for Guides in Metrology, 2008. [Online]. Available: http://www.bipm.org/en/publications/guides/gum.html.

[4] JCGM 102:2011, *Evaluation of Measurement Data--Supplement 2 to the Guide to the Expression of Uncertainty in Measurement--Extension to any number of output quantities,* Joint Committee for Guides in Metrology, 2011. [Online]. Available: http://www.bipm.org/en/publications/guides/gum.html.

[5] M. Bergoglio, A. Malengo, and D. Mari, "Analysis of interlaboratory comparisons affected by correlations of the reference standards and drift of the travelling standards," *Measurement*, vol. 44, no. 8, pp. 1461-1467, 2011. [Online]. Available: http://www.sciencedirect.com/science/article/pii/S0263224111001680.

[6] D. Ugryumova, R. Pintelon, and G. Vandersteen, "Frequency response matrix estimation from missing input-output data," *IEEE Trans. Instrum. Meas.*, vol. 64, no. 11, pp. 3124-3136, Nov. 2015.

[7] J. Sequeira, A. Tsourdos, and S. B. Lazarus, "Robust covariance estimation for data fusion from multiple sensors," *IEEE Trans. Instrum. Meas.*, vol. 60, no. 12, pp. 3833-3844, Dec. 2011.

[8] G. D. Leo, C. Liguori, and A. Paolillo, "Covariance propagation for the uncertainty estimation in stereo vision," *IEEE Trans. Instrum. Meas.*, vol. 60, no. 5, pp. 1664-1673, May 2011.
[9] P. D. Hale, A. Dienstfrey, J. C. M. Wang, D. F. Williams, A. Lewandowski, D. A. Keenan, and T. S. Clement, "Traceable waveform calibration with a covariance-based uncertainty analysis," *IEEE Trans. Instrum. Meas.*, vol. 58, no. 10, pp. 3554-3568, Oct. 2009.
[10] D. A. Humphreys, P. M. Harris, M. Rodrguez-Higuero, F. A. Mubarak, D. Zhao, and K. Ojasalo, "Principal component compression method for covariance matrices used for uncertainty propagation," *IEEE Trans. Instrum. Meas.*, vol. 64, no. 2, pp. 356-365, Feb. 2015.
[11] C. Dubois, "Covariance dans le processus d'etalonnage," in Proc. *16th International Congress of Metrology*, Paris, France, pp. 1-6, Oct. 2013.
[12] A. Ferrero, "The pillars of metrology," *IEEE Instrum. Meas. Mag.*, vol. 18, no. 6, pp. 7-11, Dec. 2015.
[13] G. Saporta, *Probabilites, Analyse des Donnees et Statistiques*, 3rd ed. Paris, France: Editions TECHNIP, 2011.

Appendix A: Covariance Expression and LO Decomposition Principle

Every measured quantity X_i can be considered, from a theoretical point of view, as the sum of a true value $x_{i,0}$ and a measurement error $E^m_{X_i}$ that can be decomposed into the sum of m errors due to the m influence quantities taken into account in the analysis of the measurement process. Of course, both $x_{i,0}$ and $E^m_{X_i}$ are unknown, so this approach is purely theoretical but useful to attain a good estimate of both.

Assuming that $E_{i,l} = 0$ if the influence quantity l does not affect the measurement process of X_i, it can be written:

$$X_i = x_{i,0} + E^m_{X_i} = x_{i,0} + \sum_{l=1}^{m} E_{i,l}. \tag{19}$$

It is assumed that errors $(E_{i,l})_{l=1\ldots m}$ do not depend on each other. The same does not apply to $(E^m_{X_i})_{i=1\ldots k}$.

The constitutive elements of the uncertainty budget of a measurement process are, generally, the systematic errors and the related standard uncertainties $(\sigma_{i,l} = \sqrt{\mathbb{V}(E_{i,l})})_{l=1\ldots m}$. Since, in general, errors may vary, whilst systematic errors do not vary by definition, measurement errors on X_i can be mathematically represented by means of their mathematical expectation $e^{sys}_{X_i} = \mathbb{E}(E^m_{X_i})$ (the systematic error) and their variance $\sigma^2_{X_i} = \mathbb{V}(E^m_{X_i})$ as:

$$e^{sys}_{X_i} = \sum_{l=1}^{m} e^{sys}_{i,l} \quad \text{and} \quad \sigma^2_{X_i} = \sum_{l=1}^{m} \sigma^2_{i,l} \tag{20}$$

Moreover, the *weight* of each influence quantity l on X_i can be written as: $w_{i,l} = \dfrac{\sigma^2_{i,l}}{\sigma^2_{X_i}}$. Of course, it is: $\sum_{l=1}^{m} w_{i,l} = 1$.

LO Decomposition Principle

This principle can be stated as follows:

Errors $(E_{i,l})_{i=1\ldots k, l=1\ldots m}$ are assumed to be ***LO decomposable*** *if independent random effects $(\Delta^{LO}_l)_{l=1\ldots m}$ exist, as well as real numbers $(\lambda_{i,l})_{i=1\ldots k, l=1\ldots m}$, called* ***LO influence coefficients****, such that the quantities $\left(E^{HO}_{i,l} = E_{i,l} - e^{sys}_{i,l} - \lambda_{i,l}\Delta^{LO}_l\right)_{i=1\ldots k}$ are pairwise independent and independent of Δ^{LO}_l for every l from 1 to m.*

The LO decomposition principle is at the very heart of the simplified covariance expression. It means that, for every influence quantity l on X_i, error $E_{i,l}$ can be decomposed into a systematic error $e^{sys}_{i,l}$, an **LO error** $E^{LO}_{i,l} = \lambda_{i,l}\Delta^{LO}_l$ with zero mean value, where $\lambda_{i,l} = 0$ if quantity l does not contribute to this error, and an **HO error** $E^{HO}_{i,l}$ with zero mean value. These errors are supposed to satisfy the usual independence assumptions.

It can be written:

$$E_{i,l} = e^{sys}_{i,l} + \lambda_{i,l}\Delta^{LO}_l + E^{HO}_{i,l} \tag{21}$$

thus taking into account that the measurement error caused on quantity X_i by the influence quantity l features a constant part $e^{sys}_{i,l}$ and two variable parts with zero mean value. The LO variable part is the one that has low opportunity to express its variations during the measurement process, while the HO part is the one that has a high opportunity to express its variations during the measurement process.

Taking into account (20) and that

$$E^{HO}_{X_i} = \sum_{l=1}^{m} E^{HO}_{i,l},$$

the measurement process can be represented by the following equation, where $x_{i,0}$ represents the unknown true value of measurand X_i:

$$X_i = x_{i,0} + e^{sys}_{X_i} + \sum_{l=1}^{m} \lambda_{i,l}\Delta^{LO}_l + E^{HO}_{X_i} \tag{22}$$

The **fraction of LO variance** of influence quantity l on X_i, which is zero by definition if $\lambda_{i,l} = 0$ is defined as:

$$\gamma_{i,l} = \frac{\mathbb{V}(E^{LO}_{i,l})}{\mathbb{V}(E_{i,l})} = \frac{\lambda^2_{i,l}\mathbb{V}(\Delta^{LO}_l)}{\sigma^2_{i,l}} \in [0,1] \tag{23}$$

The evaluation of this fraction completes the uncertainty budget for every influence quantity. To do so, the assumption of independence and bi-linearity of covariances [13] is made, so that:

$$\begin{aligned}\sigma_{X_i,X_j} &= \mathbb{C}ov\left(E^m_{X_i}, E^m_{X_j}\right) = \sum_{u=1}^{m} \lambda_{i,u} \sum_{v=1}^{m} \lambda_{j,v} \mathbb{C}ov\left(\Delta^{LO}_u, \Delta^{LO}_v\right) \\ &= \sum_{l=1}^{m} \lambda_{i,l}\lambda_{j,l}\mathbb{V}(\Delta^{LO}_l) = \sigma_{X_i}\sigma_{X_j} \sum_{l=1}^{m} sgn(\lambda_{i,l}\lambda_{j,l})\sqrt{\gamma_{i,l}\gamma_{j,l}}\sqrt{w_{i,l}w_{j,l}}\end{aligned} \tag{24}$$

where $sgn(\lambda_{i,l}\lambda_{j,l}) = 1$ if quantity l has the same effect on X_i and X_j, and $sgn(\lambda_{i,l}\lambda_{j,l}) = -1$, if it has the opposite effect. In the following, notation $\pm^{i,j}_l$ will be used for $sgn(\lambda_{i,l}\lambda_{j,l})$. The above equation shows that the covariance between X_i and X_j can be expressed as the product of the standard uncertainties of these single measured quantities, weighed by a coefficient that depends on the fraction of LO variance γ and the weight w that each influence quantity has on the measurement result.

Equality Assumption of the Fractions of LO Variances

The relative fractions of LO uncertainty that can be attributed to each influence quantity are the same for every affected quantity. In other words, for every l from 1 *to m, all non-zero* $(\gamma_{i,l})_{i=1\ldots k}$ *are equal and noted with* $\gamma_l > 0$.

This simplified assumption is justified by the fact that an LO influence reduces the range of variation of the affected quantities by the same way during the measurement process. This assumption is often verified in practice, since the fractions of LO variances are generally estimated by orders of magnitude and keep their validity when the influence factors affect different measurement processes. Except for special cases, this assumption seems more reasonable than others often based on imagination, more than on real knowledge.

Under this assumption, (24) becomes:

$$\sigma_{X_i,X_j} = \sigma_{X_i}\sigma_{X_j}\sum_{l=1}^{m}\pm_l^{i,j}\gamma_l\sqrt{w_{i,l}w_{j,l}} \qquad (25)$$

which proves (5), axiomatically considered in the previous sections.

The above derivation is based on some strong assumptions that, although quite reasonable in most industrial applications, are only assumptions and can be challenged. However, it is worth considering that the aim of an uncertainty budget is not only that of evaluating standard uncertainties, but also that of estimating the impact of phenomena that, without an accurate model, can be hardly quantified. In this respect, as also shown by the given practical examples, (25) has the great advantage of being easily applicable and provides a quick estimate of covariances in a way accurate enough for most practical problems found in the industrial practice.

Christophe Dubois (cdubois@deltamu.com) is a graduate engineer, and worked for over thirteen years in a service company specialized in computing industrial engineering. For the past five years, he has worked for Deltamu, a French consulting society in metrology. He holds the office of trainer and consultant on industrial issues in the measurement field. His main activity is to understand the industrial needs of the customers in different fields, from biology to mechanics and electrical engineering, and provide solutions.

Laurent Leblond (laurent.leblond121@orange.fr) is an expert in Industrial Statistics at the Quality Direction of the PSA Group. He got his degree in statistics in France, and the Ecole Nationale de la Statistique et de l'Administration Economique. He started his career as a research engineer at the French National Institute for Health and Medical Research, and, thirteen years ago, he joined the PSA Group. His main function is the development and implementation of references for Quality applications, and he follows and advices the functions related to the Design and Manufacturing based on Industrial Statistics in the automotive field. Metrology is a relevant part of this field. He serves also as an expert of the Statistics Committee of AFNOR and he is a Member of the French Society of Statistics.

Jean-Michel Pou (jmpou@deltamu.com) is self-taught and the founder of the French company Deltamu, which is active in the field of excellence metrology and well known for its educational activity and software development in the metrology field. He is also a member of the Metrology Committee of AFNOR, the President of the cluster of excellence Auvergne Efficience Industriel and Ambassador of the Association Alliance for the Future Industry in the French Region.

Alessandro Ferrero (Alessandro.ferrero@polimi.it) received the laurea degree in electrical engineering from Politecnico di Milano in 1978. In 1983, he joined the Dipartimento di Elettrotecnica, Politecnico di Milano, as an Assistant Professor of electrical measurements. From 1987 to 1991, he was with the University of Catania, Italy, as an Associate Professor of measurement on electrical machines and systems. From 1991 to 1994, he was with the Dipartimento di Elettrotecnica, Politecnico di Milano as an Associate Professor of electrical measurement, where he has been a Full Professor since 1994. His current research interests include uncertainty evaluation, the application of digital methods to electrical measurements and measurements on electric power systems under non-sinusoidal conditions. The results of his research activity have been published in over 250 papers in scientific journals and conference proceedings, and one book.